

International Journal of Modern Physics D
 © World Scientific Publishing Company

Non-linearity of large-scale structure formation in the Universe

Lung-Yih Chiang

*Niels Bohr Institute,
 Blegdamsvej 17, DK-2100 Copenhagen, Denmark
 chiang@nbi.dk*

In the standard picture of cosmological structure formation, the Universe we see today is evolved under the gravitational instability from tiny random fluctuations. In this talk I discuss the onset of non-linearity in the large scale structure formation of the Universe when the linear perturbation theory break downs. Using 1D Zel'dovich Approximation which provides an exact solution for density evolution, I illustrate two effects: mode spawning and mode merging and their connection to mode coupling. Those mode couplings (quadratic, cubic ...etc.) from gravitational clustering are in fact what the polyspectra (bispectrum, trispectrum ...etc.) are meant to measure.

1. Introduction

One of the main issues in cosmology is quantitative characterization of the large-scale structure of the Universe. Recent large sky surveys such as SDSS (see <http://www.sdss.org> and the publications therein) and 2dFGRS (see <http://www.mso.anu.edu.au/2dFGRS/> and the publications therein) have displayed rich structures of our Universe. In the framework of the inflation paradigm^{1,2,3,4,5}, the structure we see today is blown up from tiny inhomogeneities originated from quantum fluctuations. This picture is confirmed by cosmic microwave background (CMB) measurement of NASA WMAP mission^{6,7,8,9,10}. The homogeneity and isotropy with fluctuations only 1 part in 10^5 of CMB at the epoch of decoupling provides the strong evidence that our Universe was very smooth. Such initial density fluctuations in the simplest inflationary universe constitute a Gaussian random field.

2. Gaussian random fields and Fourier phases

Gaussian random Fields^{11,12} are useful because of its analytical simplicity. One particularly interesting property of Gaussian random fields is that the real and imaginary part of the Fourier modes are both Gaussian distributed and mutually independent, or in other words, the Fourier phases are randomly distributed between 0 and 2π . The statistical properties are then completely specified by its second-order statistics: its two-point correlation function $\xi(r)$ (with zero connected n -point correlation functions for $n > 2$), or alternatively, its power spectrum $P(k)$.

In the framework of gravitational instability, a perturbative method can be adopted at the early stage of density clustering (see e.g. Peebles 1980, Sahni and Coles 1995, Bernardeau et al. 2002). The linear perturbation theory is applicable when the density fluctuations are small i.e. its variance of density contrast $\langle \delta^2 \rangle < 1$. The statistical distribution of the density field such as originally random phases remains invariant in the linear regime, except its variance increasing with time.

The departure of the density field from the linear regime gives rise to phase coupling. Second-order statistics, such as power spectrum and two-point correlation function, throw away the fine details of the delicate pattern of cosmic structure. These details lie in the distribution of Fourier phases to which second-order statistics are blind. The evident shortcomings of $P(k)$ can be partly ameliorated by defining higher-order quantities such as the bispectrum ¹³, or correlations of $\delta(\mathbf{k})^2$ ¹⁶. Higher-order correlations and polyspectra find this information term by term so that an infinite hierarchy is required for a complete statistical characterization of the fluctuation field.

3. Fourier description

The mathematical description of an inhomogeneous Universe revolves around the dimensionless density contrast, $\delta(\mathbf{x})$, which is obtained from the spatially-varying matter density $\rho(\mathbf{x})$ via

$$\delta(\mathbf{x}) = \frac{\rho(\mathbf{x}) - \rho_0}{\rho_0}, \quad (1)$$

where \mathbf{x} is the comoving coordinate, ρ_0 is the global mean density ¹³. The two-point covariance function, which measures the excess probability over Poisson distribution between a pair with separation \mathbf{r} , is defined as

$$\xi(\mathbf{r}) = \langle \delta(\mathbf{x}) \delta(\mathbf{x} + \mathbf{r}) \rangle, \quad (2)$$

where the mean is taken over all points \mathbf{x} . It is also useful to expand the density contrast in Fourier series, in which δ is treated as a superposition of plane waves:

$$\delta(\mathbf{x}) = \sum \delta(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}). \quad (3)$$

The Fourier transform $\delta(\mathbf{k})$ is complex and therefore possesses both amplitude $|\delta(\mathbf{k})|$ and phase $\phi_{\mathbf{k}}$ where

$$\delta(\mathbf{k}) = |\delta(\mathbf{k})| \exp(i\phi_{\mathbf{k}}). \quad (4)$$

The power spectrum is defined as

$$\langle \delta(\mathbf{k}_1) \delta(\mathbf{k}_2) \rangle = (2\pi)^3 P(k) \delta^D(\mathbf{k}_1 + \mathbf{k}_2), \quad (5)$$

which is the Fourier transform of the two-point covariance function via Wiener-Khintchin theorem. We can analogously define the bispectrum as the third-order moment in Fourier space:

$$\langle \delta(\mathbf{k}_1) \delta(\mathbf{k}_2) \delta(\mathbf{k}_3) \rangle = (2\pi)^3 B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3), \quad (6)$$

which is the Fourier transform of the connected three-point covariance function.

4. The linear theory of density perturbations

Gravitational instabilities is believed to be the driving force of large-scale structure formation. When the density perturbation is small, evolution of the density contrast can be obtained analytically through the *linear perturbation theory* from 3 coupled partial differential equations. They are the linearized continuity equation,

$$\frac{\partial \delta}{\partial t} = -\frac{1}{a} \nabla_x \cdot \mathbf{v}, \quad (7)$$

the linearized Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a} \mathbf{v} = -\frac{1}{\rho a} \nabla_x p - \frac{1}{a} \nabla_x \phi, \quad (8)$$

and the linearized Poisson equation

$$\nabla_x^2 \phi = 4\pi G a^2 \rho_0 \delta. \quad (9)$$

In these equations, a is the expansion factor, p is the pressure, ∇_x denotes a derivative with respect to the comoving coordinates \mathbf{x} , $\mathbf{v} = a\dot{\mathbf{x}}$ is the peculiar velocity and $\phi(\mathbf{x}, t)$ is the peculiar gravitational potential. From Eq.(7)-(9), and if one ignores pressure forces, it is easy to obtain an equation for the evolution of δ :

$$\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\delta - 4\pi G \rho_0 \delta = 0. \quad (10)$$

For a spatially flat universe dominated by pressureless matter, $\rho_0(t) = 1/6\pi G t^2$ and Eq.(10) admits two linearly independent power-law solutions

$$\delta(\mathbf{x}, t) = b_{\pm}(t) \delta_0(\mathbf{x}), \quad (11)$$

where $\delta_0(\mathbf{x})$ is the initial density distribution, $b_+(t) \propto a(t) \propto t^{2/3}$ is the growing mode and $b_-(t) \propto t^{-1}$ is the decaying mode.

As one can see from Eq.(11), the growth depends only on time, so the growth of density distribution in the linear regime does not alter the phases.

5. The Zel'dovich Approximation

In order to examine analytically non-linear effects induced by gravitational clustering, we use the Zel'dovich approximation⁷ as a clustering scheme, which extrapolates the evolution of density perturbations into non-linear regime. This extrapolation from the linear theory follows the perturbation in particle trajectories rather than in density fields. In the Zel'dovich approximation (ZA), a particle initially placed at the Lagrangian coordinate \mathbf{q} is perturbed after a time t to an Eulerian coordinate \mathbf{x} . The displacement of the particle simply depends on the constant velocity the particle has when it is kicked off its initial position, and can be written as $b(t)\mathbf{u}(\mathbf{q})$, so that

$$\mathbf{r}(\mathbf{q}, t) = a(t)\mathbf{x}(\mathbf{q}, t) = a(t)[\mathbf{q} + b(t)\mathbf{u}(\mathbf{q})], \quad (12)$$

where \mathbf{r} is the resultant physical coordinate, $a(t)$ is the expansion factor and $b(t)$ is the growing mode $b_+(t)$ from the linear perturbation theory. According to this prescription, each particle moves with a constant velocity along a ballistic trajectory, which resembles Newtonian inertial motion. Note that the peculiar velocity according to the ZA is $a\dot{\mathbf{x}} = a(t)\dot{b}(t)\mathbf{u}(\mathbf{q})$. For an irrotational flow, $\mathbf{u}(\mathbf{q})$ can be expressed as a gradient of some velocity potential $-\nabla\Phi_0(\mathbf{q})$.

We focus particularly on the applications of one-dimensional ZA. The ZA in 1D provides an exact solution of density evolution in that the evolution of planar collapse from the ZA has the same solution as from the Poisson equation¹⁸ until shell-crossing. The ZA in 1D is simplified to

$$x(q, t) = q + b(t)u(q) = q - b(t)\frac{d\Phi_0(q)}{dq}. \quad (13)$$

The density contrast can be derived from the conservation of mass $\rho dx = \rho_0 dq$:

$$\delta = \frac{\rho}{\rho_0} - 1 = \left(\frac{\partial x}{\partial q} \right)^{-1} - 1 = \left[1 - b(t) \frac{d^2\Phi_0(q)}{dq^2} \right]^{-1} - 1 \quad (14)$$

$$= \sum_n b^n(t) \left(\frac{d^2\Phi_0}{dq^2} \right)^n. \quad (15)$$

The velocity potential $\Phi_0(q)$ can always be disintegrated as

$$\Phi_0(q) = \sum_i A_i \cos(\lambda_i q + \alpha_i). \quad (16)$$

6. Mode spawning and mode merging

Upon using 1D ZA, we firstly choose as a toy model the velocity potential^a

$$\Phi_0(q) = -A_1 \cos(\lambda_1 q + \alpha_1) - A_2 \cos(\lambda_2 q + \alpha_2). \quad (17)$$

When $b(t) \ll 1$, the density contrast can be expressed in terms of only the first order from Eq.(15)

$$\begin{aligned} \delta \simeq \delta^{(1)} &= b(t) \left(\frac{d^2\Phi_0}{dq^2} \right) \\ &= a_1 \cos(\lambda_1 q + \alpha_1) + a_2 \cos(\lambda_2 q + \alpha_2) \\ &\equiv a_1 \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix} + a_2 \begin{pmatrix} \lambda_2 \\ \alpha_2 \end{pmatrix}, \end{aligned} \quad (18)$$

where the round brackets hereafter denote cosine functions¹⁹ in order to show clearly the harmonic relationship (see below), $a_1 = b(t)A_1\lambda_1^2$ and $a_2 = b(t)A_2\lambda_2^2$. After Fourier transform in Lagrangian coordinate we have only two modes with wavenumbers λ_1 and λ_2 , and phases α_1 and α_2 , respectively. So these are the only

^aWe put minus sign for both cosine functions in the velocity potential to make the analytical form neat, assuming both A_1 and A_2 positive, otherwise the phases would have a shift by π .

Fourier modes the moment the clustering process begins (which we call the “parent modes”). It is clear that the first order does not display phase coupling. When the second-order term becomes comparable,

$$\begin{aligned}\delta \simeq \delta^{(1)} + \delta^{(2)} &= b(t) \left(\frac{d^2 \Phi_0}{dq^2} \right) + b^2(t) \left(\frac{d^2 \Phi_0}{dq^2} \right)^2 \\ &= \frac{a_1^2 + a_2^2}{2} + a_1 \begin{pmatrix} \lambda_1 \\ \alpha_1 \end{pmatrix} + a_2 \begin{pmatrix} \lambda_2 \\ \alpha_2 \end{pmatrix} \\ &\quad + a_1^2 \begin{pmatrix} 2\lambda_1 \\ 2\alpha_1 \end{pmatrix} + a_2^2 \begin{pmatrix} 2\lambda_2 \\ 2\alpha_2 \end{pmatrix} + a_1 a_2 \begin{pmatrix} \lambda_1 + \lambda_2 \\ \alpha_1 + \alpha_2 \end{pmatrix} + a_1 a_2 \begin{pmatrix} \lambda_1 - \lambda_2 \\ \alpha_1 - \alpha_2 \end{pmatrix}. \quad (19)\end{aligned}$$

Since the density contrast in 1D ZA can be expressed as a power series of $d^2 \Phi_0 / dq^2$ in Eq. (15), the second-order term reflects the quadratic density fields^{20,21}. This second-order term spawns modes with wavenumbers that are from combinations of any 2 wavenumbers from the parent ones, i.e. Fourier modes with wavenumbers $2\lambda_1$, $2\lambda_2$, $\lambda_1 + \lambda_2$ and $\lambda_1 - \lambda_2$ spawned from combination of λ_1 and λ_2 . One important feature of quadratic density fields is that the phases of the spawned modes follow the same kind of harmonic relationship as the spawned wavenumbers, which we call hereafter “wavenumber-phase harmonic relation”. Such relation subsequently forms phase associations between Fourier modes and is crucial for bispectrum analysis.

We can generalize the velocity potential in 1D ZA as a sum of cosine functions $\Phi_0 = -\sum_i A_i \cos(\lambda_i q + \alpha_i)$. The density contrast of the first order and that up to the second order now become, respectively,

$$\delta^{(1)} = \sum_i a_i \cos(\lambda_i + \alpha_i) \equiv \sum_i a_i \begin{pmatrix} \lambda_i \\ \alpha_i \end{pmatrix}, \quad (20)$$

where $a_i = b(t) A_i \lambda_i^2$, and

$$\delta^{(1)} + \delta^{(2)} = \sum_i a_i \begin{pmatrix} \lambda_i \\ \alpha_i \end{pmatrix} + \sum_{jk} \frac{a_j a_k}{2} \left[\begin{pmatrix} \lambda_j - \lambda_k \\ \alpha_j - \alpha_k \end{pmatrix} + \begin{pmatrix} \lambda_j + \lambda_k \\ \alpha_j + \alpha_k \end{pmatrix} \right]. \quad (21)$$

If primordial Gaussianity is assumed, the phases α_i of the initial velocity potential are uniformly random and independently distributed between 0 and 2π . Below we categorize the effects induced by quadratic density fields.

- **Mode spawning** : In Eq.(21) beyond the first order, new modes are spawned from the $\delta^{(2)}$ and correlated phases are created following the spawned modes. The wavenumbers of the spawned modes are formed from combination of any 2 wavenumbers of the first order (the same as the toy model) and the phases follow the *wavenumber-phase harmonic relation*. Some terms may appear even at the earliest stage (when $b(t) A_j A_k (\lambda_j \lambda_k / \lambda_i)^2 > 2A_i$). Those are terms usually involving high-frequency modes, i.e. high λ_j or λ_k . Such phase coupling can manifest

itself through phase mapping ²². For example, a short sequence of modes is formed with a constant difference in wavenumber $\Delta k = \lambda_k$:

$$\frac{a_j a_k}{2} \left(\frac{\lambda_j - \lambda_k}{\alpha_j - \alpha_k} \right), a_j \left(\frac{\lambda_j}{\alpha_j} \right), \frac{a_j a_k}{2} \left(\frac{\lambda_j + \lambda_k}{\alpha_j + \alpha_k} \right), \quad (22)$$

where the middle mode is taken from the $\delta^{(1)}$. Due to the wavenumber-phase harmonic relation, this phase sequence has a constant phase difference $\Delta\phi = \alpha_k$ and can be mapped along a line parallel to the diagonal through phase mapping technique. Such coupling between modes with large Δk is discussed in Chiang, Coles & Naselsky ²².

- **Mode merging :** The newly-spawned modes from the second order are not all independent but could merge with modes of the same wavenumbers in the first order. Take the following two modes from Eq.(21) as an example, one from the $\delta^{(1)}$ and the other from the $\delta^{(2)}$:

$$a_i \left(\frac{\lambda_i}{\alpha_i} \right), \frac{a_j a_k}{2} \left(\frac{\lambda_j - \lambda_k}{\alpha_j - \alpha_k} \right). \quad (23)$$

If $\lambda_j - \lambda_k = \lambda_i$, i.e. the wavenumber of the newly-spawned mode coincides with the one in the first order, these two modes merge to form a new mode:

$$\sqrt{a_i^2 + \frac{a_j^2 a_k^2}{4} + a_i a_j a_k \cos(\alpha_j - \alpha_k - \alpha_i)} \left(\frac{\lambda_i}{\alpha_m} \right), \quad (24)$$

where

$$\alpha_m = \arctan \frac{a_i \sin \alpha_j + a_j a_k \sin(\alpha_j - \alpha_k)/2}{a_i \cos \alpha_j + a_j a_k \cos(\alpha_j - \alpha_k)/2} \quad (25)$$

where the amplitude is modulated and the phase is shifted. Such modulation in amplitudes and phases proceeds gradually and the consequence is that the wavenumber-phase harmonic relation is broken. Thus not all spawned modes will enjoy the harmonic relation. Note that in this case the evolution can still be at very early stage $b(t) \ll 1$ as long as $\lambda_j \lambda_k / \lambda_i \gg (A_i / A_j A_k)^{1/2}$.

7. Full treatment of the 1D ZA

To account for the interaction of the parent modes leading to mode merging, we can give a full treatment of 1D ZA by starting from Eq. (14). We have

$$\begin{aligned} \delta_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\left(\frac{dx}{dq} \right)^{-1} - 1 \right] e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} dq = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\{-ik[q + b(t)u(q)]\} dq, \end{aligned} \quad (26)$$

where $u(q) = -d\Phi_0(q)/dq$. We can give the same treatment to Eq. (26) as Eq. (15) by expressing it in terms of a power series of $b(t)$:

$$\delta_k = \sum_{p=0}^{\infty} \frac{(-ikb)^p}{p!} \frac{1}{2\pi} \int_{-\pi}^{\pi} dq e^{-ikq} u(q)^p. \quad (27)$$

Following Soda & Suto ²³, we can denote the Fourier transform of $u(q)$ as u_k

$$u_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq u(q) e^{-ikq}, \quad (28)$$

so the Fourier modes δ_k can be expressed as

$$\begin{aligned} \delta_k = & -ikbu_k + \frac{(-ikb)^2}{2!} \sum_{k_1, k_2} u_{k_1} u_{k_2} \delta^D[k - (k_1 + k_2)] \\ & + \frac{(-ikb)^3}{3!} \sum_{k_1, k_2, k_3} u_{k_1} u_{k_2} u_{k_3} \delta^D[k - (k_1 + k_2 + k_3)] + \dots, \end{aligned} \quad (29)$$

where δ^D denotes Dirac-delta function. For $u(q) = -d\Phi_0/dq = -\sum_j A_j \lambda_j \sin(\lambda_j q + \alpha_j)$ from the velocity potential $\Phi_0 = -\sum_j A_j \cos(\lambda_j q + \alpha_j)$,

$$u_k = -\frac{1}{2i} \sum_j A_j \lambda_j e^{i\alpha_j} \delta^D(\lambda_j - k) = -\frac{A_k \lambda_k}{2i} e^{i\alpha_k} = \frac{A_k \lambda_k}{2} e^{i(\pi/2 + \alpha_k)}, \quad (30)$$

and u_{-k} is simply its complex conjugate.

Equation (29) and (30) provide us the insight into mode spawning from the initial potential field $\Phi_0(q)$ and mode merging in gravitational clustering. The Dirac-delta functions dictate the spawning of Fourier modes, both the amplitudes and phases. If, for any Fourier mode k , there is only one non-zero term in Eq.(29), either from the parent modes in the $\Phi_0(q)$ (i.e. from the 1st-order term $-ikbu_k$) or any combination from parent modes such that there is only one non-zero term in Eq.(29) triggered by the non-zero δ^D , its phase α_k shall not change and the amplitude grow steadily with $b(t)^n$ (according to the order of this combination). If, on the other hand, there are more than one non-zero term in Eq.(29), indicating there is more than one set of parent-mode combination for the wavenumber k [e.g. Eq. (23)], they will interact so that the phase α_k will change accordingly.

Take a toy model as an example, where the initial field has only two modes: 7 and 11. For the 1st-order term of Eq.(29), only u_7 and u_{11} exist as the parent modes. For the 2nd-order term, those which are non-zero for δ^D are $k = 7+7, 7+11, 11-7, 11+11$; for the 3rd-order term, those which are non-zero are $k = 7+7+7, 11+11+11, 7+7-11, 7+7+11, 11+11-7$ and $11+11+7$...etc.. The corresponding phases of these spawned modes come from multiplication of u_7 and u_{11} and do not interact with each other.

However, if we have another toy model where the initial field has two modes, 7 and 14, again the parent modes are u_7 and u_{14} . Now for the Fourier mode $\delta_{k=14}$, for example, we have not only $b(t) u_{14}$ from the 1st-order term, but also $b(t)^2 u_7^2 \delta^D[k - (7+7)]$ from the 2nd-order term, or even higher terms such as the 4th-order $b(t)^4 u_7^2 u_{14}^2 \delta^D[k - (14+14-7-7)]$ and the 5th-order $b(t)^5 u_7^4 u_{14} \delta^D[k - (7+7+7+7-14)]$...etc.. The evolution for $\delta_{k=14}$ is therefore from the vector addition of all these terms. Of course the higher order terms come into force and dominate at later times. One can then imagine that for an initial density distribution which has continuous Fourier modes, the modes spawned shall immediately

interact with some parent modes, thereby complicating the phase and amplitude evolution in Fourier domain.

8. Conclusion

In this talk, I have used 1D ZA to illustrate some non-linear effects induced by gravitational clustering: mode spawning and mode coupling. The onset of non-linearity can be illustrated by expanding the density contrast in terms of power series of $b(t)$. The second-order of such expansion reflects quadratic density fields, from which modes are spawned with phases following a special wavenumber-phase harmonic relation. The wavenumber-phase harmonic relation holds as long as the wavenumbers of parent modes do not interact. In a full treatment of 1D ZA, such mode spawning and coupling take place not only during the departure from the linear regime, but also the entire gravitational clustering process. The complexity of Fourier amplitude and phase evolution in gravitational clustering comes from continuous interactions between mode merging and coupling with the spawned modes.

References

1. A. A., Starobinsky., *Pis'ma Zh. Eksp. Teor. Fiz.*, 30, 719 (1979)
2. A. H., Guth., *Phys. Rev. D*, 23, 347 (1981)
3. A. H., Guth., S.-Y., Pi., *Phys. Rev. Lett.*, 49, 1110 (1982)
4. A., Albrecht., P. J., Steinhardt., *Phys. Rev. Lett.*, 48, 1220 (1982)
5. A. D., Linde., *Phys. Lett. B*, 108, 389 (1982)
6. C. L., Bennett., et al., *Astrophys. J.*, 583, 1 (2003)
7. C. L., Bennett., et al., *Astrophys. J. Supp.*, 148, 1 (2003)
8. G., Hinshaw., et al., *Astrophys. J. Supp.*, 148, 135 (2003)
9. E., Komatsu., et al., *Astrophys. J. Supp.*, 148, 119 (2003)
10. D. N., Spergel., et al., *Astrophys. J. Supp.*, 148, 175 (2003)
11. R. J., Adler., 1981, The Geometry of Random Fields, John Wiley & Sons, New York
12. J. M., Bardeen., J. R., Bond., N., Kaiser., A. S., Szalay., *Astrophys. J.*, 304, 15 (1986)
13. P. J. E., Peeble., 1980, The Large-scale structure of the Universe, Princeton University Press
14. V., Sahni., P., Coles., *Phys. Rep.*, 262, 1 (1995)
15. F., Bernardeau., S., Colombi., E., Gaztanaga., R., Scoccimarro., *Phys. Rep.*, 367, 1 (2002)
16. A. J., Stirling., J. A., Peacock., *Mon. Not. R. Ast. Soc.*, 283, L99 (1996)
17. Ya. B., Zel'dovich., *Astro. & Astrophys.*, 5, 84 (1970)
18. T., Buchert., *Mon. Not. R. Ast. Soc.*, 254, 729 (1992)
19. P., Coles., L.-Y., Chiang., 2001, in A.J., Banday., S., Zaroubi., M., Bartelmann., eds., Proc. of the MPA/ESO/MPE Workshop "Mining the Sky" held at Garching, Germany, July 31 - August 4, 2000, Springer-Verlag series "ESO Astrophysics Symposia" (astro-ph/0010521)
20. P., Coles., J. D., Barrow., *Mon. Not. R. Ast. Soc.*, 228, 407 (1987)
21. P.I.R., Watts., P., Coles., *Mon. Not. R. Ast. Soc.*, 338, 806 (2002)
22. L.-Y., Chiang., P., Coles., P. D., Naselsky., *Mon. Not. R. Ast. Soc.*, 337, 488 (2002)
23. J., Soda., Y., Suto., *Astrophys. J.*, 396, 379 (1992)